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Gaudin magnet with boundary and generalized Knizhnik–Zamolodchikov equation

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Abstract. We consider the boundary quantum inverse scattering method established by Sklyanin. The Gaudin magnet with boundary is diagonalized by taking a quasi-classical limit of the inhomogeneous lattice. Using the method proposed by Babujian, the integral representation for the solution of the B-type Knizhnik–Zamolodchikov equation is explicitly constructed.

1. Introduction

The algebraic Bethe ansatz (ABA) method for the open-boundary spin chain is considered to construct the integral solution of the Knizhnik–Zamolodchikov type dfferential equation. This technique was first studied by Sklyanin [1] to treat the spin-1/2 XXZ spin chain, and has been widely applied to other systems [2–4]. Although the open-boundary spin chain was solved in [5] by use of the coordinate Bethe ansatz method, the ABA approach reveals the quantum group structure of this system [6, 7]. In this formulation both the R-matrix and K-matrix play crucial roles. These matrices are solutions of the Yang–Baxter and reflection equations, respectively. We only treat the rational and the trigonometric solution of these equations; the elliptic K-matrix as a general solution of the reflection equation has been investigated [8, 9].

In the first part of this paper we construct the eigenstate of the boundary Gaudin magnet by taking a quasi-classical limit of the transfer matrix for the inhomogeneous open spin chain. The Gaudin magnet has its origins in [10, 11] as an integrable spin system with long-range interaction. The Hamiltonian is given as a solution of the classical Yang–Baxter equation [12, 13], and is closely connected with the notion of the separation of variables [14–17]. The 'functional Bethe ansatz' method was also applied to the boundary Gaudin magnet [18].

In the second part, the Knizhnik–Zamolodchikov type differential equation is studied. Recently, an interesting structure of the Gaudin magnet has been revealed; the integral solution of the Knizhnik–Zamolodchikov (KZ) equation can be obtained in terms of the Bethe eigenstate of the Gaudin magnet [19–22]. The KZ equation is a set of differential equations which is satisfied by the correlation functions of the WZNW model [23]. The integral formula for the solution of the KZ equation has been derived based either on the theory of the hypergeometric function [24, 25], or on the Wakimoto construction [26]. The relationship between the Gaudin magnet and the KZ equation gives an insight into the

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structure of the integral solution. Making use of this technique for the Gaudin magnet with boundary, we shall construct the integral representation for solutions of the B-type KZ equation.

This paper is organized as follows. In section 2 the ABA method for the open-boundary spin chain is reviewed. The transfer matrix for su(2) spin-1/2 XXZ chain is constructed in terms of the R-matrix and the K-matrix. We show in section 3 that the Hamiltonian of the boundary Gaudin magnet appears by taking a quasi-classical limit of the transfer matrix; this completes the diagonalization of the boundary Gaudin magnet. Based on the 'off-shell Bethe ansatz method' of Babujian, the explicit integral solution for the B-type KZ equation is obtained in section 4. Section 5 is devoted to a summary and discussion.

In the following we use the Pauli spin matrices as a two-dimensional representation for the su(2) Lie algebra:

$$\sigma^x = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \qquad \sigma^y = \begin{pmatrix} & -\mathrm{i} \\ \mathrm{i} & \end{pmatrix} \qquad \sigma^z = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}. \tag{1.1}$$

The creation–annihilation operators $\sigma^{\pm}$ are also used:

$$\sigma^{\pm} = \frac{1}{2}\left(\sigma^x \pm \mathrm{i}\sigma^y\right). \tag{1.2}$$

We remark that these spin operators act on the Hilbert space $V = \mathbb{C}^2$.

2. The boundary quantum inverse scattering method

We briefly review the boundary quantum inverse scattering method (QISM) of Sklyanin [14]. The basic notion is the quantum R-matrix satisfying the Yang–Baxter equation (YBE) (figure 1):

$$R_{12}(u)R_{13}(u+v)R_{23}(v) = R_{23}(v)R_{13}(u+v)R_{12}(u). \tag{2.1}$$

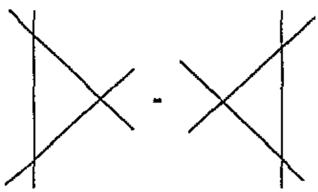

Figure 1. The Yang–Baxter equation.

Here R_{jk} signifies the matrix on $V \otimes V \otimes V$ acting on the jth and kth spaces and as an identity on the other space. The variables u and v are called the spectral parameters. As a solution of YBE (2.1), we use the R-matrix for the six-vertex model [27, 28]:

$$R(u) = \begin{pmatrix} 1 & & & \\ & b(u) & c(u) & \\ & c(u) & b(u) & \\ & & & 1 \end{pmatrix} \tag{2.2}$$

where

$$b(u) = \frac{\operatorname{sh} u}{\operatorname{sh}(u+\eta)} \qquad c(u) = \frac{\operatorname{sh} \eta}{\operatorname{sh}(u+\eta)}. \tag{2.3}$$

In the rational limit, these functions reduce to

$$b(u) = \frac{u}{u+\eta} \qquad c(u) = \frac{\eta}{u+\eta}$$

and the R-matrix is simply written as $R(u) = (u + \eta P)/(u + \eta)$. We note that the R-matrix depends not only on the spectral parameter u but on the 'deformation parameter' η.

The R-matrix defined in (2.2) has the following properties.

(i) regularity:

$$R(u = 0) = P \tag{2.4}$$

(ii) the quasi-classical condition:

$$R(u)|_{\eta \to 0} = 1 \tag{2.5}$$

(iii) unitarity:

$$R(u)R(-u) = 1. \tag{2.6}$$

Using the R-matrix satisfying YBE (2.1) we can define the monodromy matrix $T(u)$ for the inhomogeneous N-site spin chain [29] (figure 2) by

$$T_0(u) = R_{0N}(u - z_N) \cdots R_{02}(u - z_2)R_{01}(u - z_1) \equiv \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}. \tag{2.7}$$

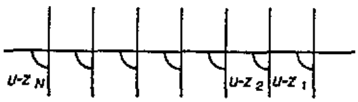

Figure 2. The monodromy matrix $T(u)$.

Here the operator matrix elements $A(u)$, $B(u)$, $C(u)$, and $D(u)$ act on the full Hilbert space $V^{\otimes N}$. Due to the additive property of the spectral parameters, the YBE also holds for the inhomogeneous lattice and we have

$$R_{12}(u - v)(T(u) \otimes 1)(1 \otimes T(v)) = (1 \otimes T(v))(T(u) \otimes 1)R_{12}(u - v). \tag{2.8}$$

This relation gives us the commutation relations between the operator matrix elements $A(u)$, $B(u)$, $C(u)$, and $D(u)$. Some of them are given as

$$[B(u), B(v)] = 0 \tag{2.9a}$$

$$A(u)B(v) = \frac{1}{b(v-u)}B(v)A(u) - \frac{c(v-u)}{b(v-u)}B(u)A(v) \tag{2.9b}$$

$$D(u)B(v) = \frac{1}{b(u-v)}B(v)D(u) - \frac{c(u-v)}{b(u-v)}B(u)D(v) \tag{2.9c}$$

$$[B(u), C(v)] = \frac{c(u-v)}{b(u-v)}(D(v)A(u) - D(u)A(v)). \tag{2.9d}$$

To construct the eigenstates of our system, we use the vacuum state $|\Omega\rangle$:

$$|\Omega\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_1 \otimes \cdots \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}_N. \tag{2.10}$$

It is easy to check that the vacuum state $|\Omega\rangle$ is the eigenstate of operators $A(u)$ and $D(u)$, and annihilated by $C(u)$:

$$A(u)|\Omega\rangle = |\Omega\rangle \tag{2.11a}$$

$$D(u)|\Omega\rangle = \prod_k^N \frac{\operatorname{sh}(u - z_k)}{\operatorname{sh}(u - z_k + \eta)}|\Omega\rangle \tag{2.11b}$$

$$C(u)|\Omega\rangle = 0. \tag{2.11c}$$

We note that the Bethe state, a sum of spin waves [30], can be generated by operator $B(u)$ acting on the vacuum state $|\Omega\rangle$.

To formulate the spin chain with open boundary, we further introduce the so-called 'boundary K-matrices' which satisfy the reflection equations (figure 3):

$$R_{21}(u-v)(K_-(u)\otimes 1)R_{12}(u+v)(1\otimes K_-(v)) \\ = (1\otimes K_-(v))R_{21}(u+v)(K_-(u)\otimes 1)R_{12}(u-v) \tag{2.12}$$

$$R_{21}(-u+v)(K_+^t(u)\otimes 1)R_{12}(-u-v-2\eta)(1\otimes K_+^t(v)) \\ = (1\otimes K_+^t(v))R_{21}(-u-v-2\eta)(K_+^t(u)\otimes 1)R_{12}(-u+v). \tag{2.13}$$

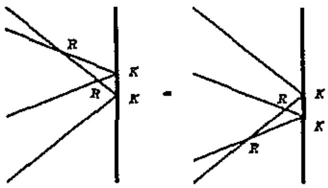

Figure 3. The reflection equation.

For the R-matrix in (2.2) the reflection equations have been solved as

$$K_-(u) = K(u, \xi_-) \tag{2.14}$$

$$K_+(u) = K(u+\eta, \xi_+) \tag{2.15}$$

where matrix $K(u,\xi)$ is diagonal [31]:

$$K(u,\xi) = \begin{pmatrix} \operatorname{sh}(u+\xi) & \\ & -\operatorname{sh}(u-\xi) \end{pmatrix}. \tag{2.16}$$

With the K-matrices satisfying the reflection equation, we can define the Yang–Baxter operator $U(u)$ as

$$U(u) = T(u)K_-(u)T^{-1}(-u) \equiv \begin{pmatrix} \mathcal{A}(u) & \mathcal{B}(u) \\ \mathcal{C}(u) & \mathcal{D}(u) \end{pmatrix} \tag{2.17}$$

where the operator $T(u)$ includes the inhomogeneity $\{z_j\}$ (2.7). Note that the operators $\mathcal{A}(u) \sim \mathcal{D}(u)$ also act on the Hilbert space $V^{\otimes N}$. It is easy to check that the operator $U(u)$ satisfies the boundary YBE:

$$R_{21}(u-v)(U(u)\otimes 1)R_{12}(u+v)(1\otimes U(v)) \\ = (1\otimes U(v))R_{21}(u+v)(U(u)\otimes 1)R_{12}(u-v). \tag{2.18}$$

The transfer matrix for the open-boundary problem is defined with the Yang–Baxter operator $U(u)$ and the reflection matrix $K_+(u)$ as (figure 4)

$$t(u) = \operatorname{Tr}_0 K_+(u)U(u). \tag{2.19}$$

This transfer matrix $t(u)$ forms a one-parameter commuting family

$$[t(u), t(v)] = 0 \tag{2.20}$$

and the Hamiltonian of the inhomogeneous XXZ spin chain with open boundary is given by

$$\mathcal{H} = \frac{\mathrm{d}}{\mathrm{d}u}\log t(u)\bigg|_{u=0}. \tag{2.21}$$

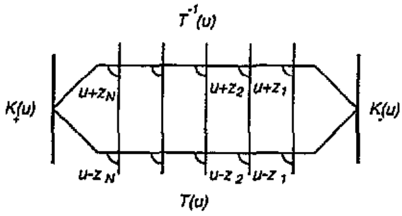

Figure 4. The transfer matrix $t(u)$ for the open-boundary spin chain.

Now the problem is to diagonalize the transfer matrix $t(u)$ (2.19). The boundary YBE (2.18) can be rewritten in terms of the operator matrix elements, $\mathcal{A}(u)$, $\mathcal{B}(u)$, $\mathcal{C}(u)$, and $\mathcal{D}(u)$:

$$[\mathcal{B}(u), \mathcal{B}(v)] = 0 \tag{2.22a}$$

$$\mathcal{A}(u)\mathcal{B}(v) = \frac{\operatorname{sh}(u+v)\operatorname{sh}(u-v-\eta)}{\operatorname{sh}(u+v+\eta)\operatorname{sh}(u-v)}\mathcal{B}(v)\mathcal{A}(u) + \frac{\operatorname{sh}(u+v)\operatorname{sh}\eta}{\operatorname{sh}(u+v+\eta)\operatorname{sh}(u-v)}\mathcal{B}(u)\mathcal{A}(v) - \frac{\operatorname{sh}\eta}{\operatorname{sh}(u+v+\eta)}\mathcal{B}(u)\mathcal{D}(v) \tag{2.22b}$$

$$\mathcal{D}(u)\mathcal{B}(v) = -\frac{2\operatorname{sh}^2\eta\operatorname{ch}\eta}{\operatorname{sh}(u-v)\operatorname{sh}(u+v+\eta)}\mathcal{B}(v)\mathcal{A}(u) + \frac{\operatorname{sh}\eta\operatorname{sh}(u-v+2\eta)}{\operatorname{sh}(u+v+\eta)\operatorname{sh}(u-v)}\mathcal{B}(u)\mathcal{A}(v) + \frac{\operatorname{sh}(u-v+\eta)\operatorname{sh}(u+v+2\eta)}{\operatorname{sh}(u-v)\operatorname{sh}(u+v+\eta)}\mathcal{B}(v)\mathcal{D}(u) - \frac{\operatorname{sh}\eta\operatorname{sh}(u+v+2\eta)}{\operatorname{sh}(u-v)\operatorname{sh}(u+v+\eta)}\mathcal{B}(u)\mathcal{D}(v). \tag{2.22c}$$

For convenience we shall use the operator

$$\widehat{\mathcal{D}}(u) = \mathcal{D}(u)\operatorname{sh}(2u+\eta) - \mathcal{A}(u)\operatorname{sh}\eta. \tag{2.23}$$

The commutation relations (2.22a) reduce to the forms

$$\mathcal{A}(u)\mathcal{B}(v) = \frac{\operatorname{sh}(u+v)\operatorname{sh}(u-v-\eta)}{\operatorname{sh}(u+v+\eta)\operatorname{sh}(u-v)}\mathcal{B}(v)\mathcal{A}(u) + \frac{\operatorname{sh}\eta\operatorname{sh}(2v)}{\operatorname{sh}(u-v)\operatorname{sh}(2v+\eta)}\mathcal{B}(u)\mathcal{A}(v) - \frac{\operatorname{sh}\eta}{\operatorname{sh}(u+v+\eta)\operatorname{sh}(2v+\eta)}\mathcal{B}(u)\widehat{\mathcal{D}}(v) \tag{2.24a}$$

$$\widehat{\mathcal{D}}(u)\mathcal{B}(v) = \frac{\operatorname{sh}(u-v-\eta)\operatorname{sh}(u+v+\eta)}{\operatorname{sh}(u-v)\operatorname{sh}(u+v+\eta)}\mathcal{B}(v)\widehat{\mathcal{D}}(u) - \frac{\operatorname{sh}\eta\operatorname{sh}(2u+2\eta)}{\operatorname{sh}(2v+\eta)\operatorname{sh}(u-v)}\mathcal{B}(u)\widehat{\mathcal{D}}(v) + \frac{\operatorname{sh}\eta\operatorname{sh}(2v)\operatorname{sh}(2u+2\eta)}{\operatorname{sh}(2v+\eta)\operatorname{sh}(u+v+\eta)}\mathcal{B}(u)\mathcal{A}(v). \tag{2.24b}$$

Note the difference of the operator algebra between (2.9a) and (2.24a) and note that the transfer matrix $t(u)$ in (2.19) is written in terms of operators $\mathcal{A}(u)$ and $\widehat{\mathcal{D}}(u)$ as

$$t(u) = \frac{\operatorname{sh}(2u+2\eta)\operatorname{sh}(u+\xi_+)}{\operatorname{sh}(2u+\eta)}\mathcal{A}(u) - \frac{\operatorname{sh}(u+\eta-\xi_+)}{\operatorname{sh}(2u+\eta)}\widehat{\mathcal{D}}(u). \tag{2.25}$$

Let us see the action of the operators $\mathcal{A}(u)$, $\mathcal{C}(u)$, and $\widehat{\mathcal{D}}(u)$ for the vacuum state $|\Omega\rangle$ (2.10). The property of the monodromy matrix (2.7) is useful:

$$T(u)\sigma^y T^t(u-\eta)\sigma^y = \operatorname{qdet} T(u) \tag{2.26}$$

where qdet means the quantum-determinant [32],

$$\begin{aligned}\operatorname{qdet} T(u) &= A(u)D(u-\eta) - B(u)C(u-\eta)\\ &= D(u)A(u-\eta) - C(u)B(u-\eta)\\ &= A(u-\eta)D(u) - C(u-\eta)B(u)\\ &= D(u-\eta)A(u) - B(u-\eta)C(u).\end{aligned}$$

Identity (2.26) can be checked by using the commutation relations (2.8). With this property and (2.11*a*) we can see that the vacuum state $|\Omega\rangle$ is also the eigenstate of operators $\mathcal{A}(u)$ and $\widehat{\mathcal{D}}(u)$ and annihilated by $\mathcal{C}(u)$:

$$\mathcal{A}(u)|\Omega\rangle = a(u)|\Omega\rangle \tag{2.27a}$$

$$\widehat{\mathcal{D}}(u)|\Omega\rangle = \widehat{d}(u)|\Omega\rangle \tag{2.27b}$$

$$\mathcal{C}(u)|\Omega\rangle = 0. \tag{2.27c}$$

Here we have defined the eigenvalue functions $a(u)$ and $\widehat{d}(u)$ as

$$a(u) = \operatorname{sh}(u+\xi_-) \tag{2.28}$$

$$\widehat{d}(u) = -\operatorname{sh}(2u)\operatorname{sh}(u-\xi_-+\eta)\left[\prod_k^N \frac{\operatorname{sh}(u-z_k)\operatorname{sh}(u+z_k)}{\operatorname{sh}(u-z_k+\eta)\operatorname{sh}(u+z_k+\eta)}\right]. \tag{2.29}$$

We shall diagonalize the transfer matrix $t(u)$ (2.25) for the open-boundary spin chain in terms of the Bethe state

$$\Psi(v) \equiv \prod_{\alpha=1}^{M} \mathcal{B}(v_\alpha)|\Omega\rangle. \tag{2.30}$$

Using the commutation relations (2.24*a*) between operators $\mathcal{A}(u)$, $\mathcal{B}(u)$ and $\widehat{\mathcal{D}}(u)$, we obtain

$$t(u)\Psi(v) = \Lambda(u)\Psi(v) + \sum_{\alpha=1}^{M} F_\alpha \Psi_\alpha(u). \tag{2.31}$$

Here the functions $\Lambda(u)$ and F_α are the so-called 'wanted' and 'unwanted' terms, respectively, and have the forms

$$\begin{aligned}\Lambda(u) = {}& \frac{\operatorname{sh}(2u+2\eta)\operatorname{sh}(u+\xi_+)}{\operatorname{sh}(2u+\eta)}\left[\prod_\alpha^M \frac{\operatorname{sh}(u-v_\alpha-\eta)\operatorname{sh}(u+v_\alpha)}{\operatorname{sh}(u-v_\alpha)\operatorname{sh}(u+v_\alpha+\eta)}\right]a(u)\\ & -\frac{\operatorname{sh}(u-\xi_++\eta)}{\operatorname{sh}(2u+\eta)}\left[\prod_\alpha^M \frac{\operatorname{sh}(u-v_\alpha+\eta)\operatorname{sh}(u+v_\alpha+2\eta)}{\operatorname{sh}(u-v_\alpha)\operatorname{sh}(u+v_\alpha+\eta)}\right]\widehat{d}(u)\end{aligned} \tag{2.32}$$

$$\begin{aligned}F_\alpha = {}& \left\{\left[\prod_{\beta\neq\alpha}^M \frac{\operatorname{sh}(v_\alpha-v_\beta-\eta)\operatorname{sh}(v_\alpha+v_\beta)}{\operatorname{sh}(v_\alpha-v_\beta)\operatorname{sh}(v_\alpha+v_\beta+\eta)}\right]a(v_\alpha)\operatorname{sh}(2v_\alpha)\operatorname{sh}(v_\alpha+\xi_+)\right.\\ & \left. -\left[\prod_{\beta\neq\alpha}^M \frac{\operatorname{sh}(v_\alpha-v_\beta+\eta)\operatorname{sh}(v_\alpha+v_\beta+2\eta)}{\operatorname{sh}(v_\alpha-v_\beta)\operatorname{sh}(v_\alpha+v_\beta+\eta)}\right]\widehat{d}(v_\alpha)\operatorname{sh}(\xi_+-v_\alpha-\eta)\right\}\\ & \times\frac{\operatorname{sh}(2u+2\eta)\operatorname{sh}\eta}{\operatorname{sh}(2v_\alpha+\eta)\operatorname{sh}(u-v_\alpha)\operatorname{sh}(u+v_\alpha+\eta)}\end{aligned} \tag{2.33}$$

$$\Psi_\alpha(u) = \mathcal{B}(u)\prod_{\beta\neq\alpha}^{M} \mathcal{B}(v_\beta)|0\rangle. \tag{2.34}$$

Relation (2.31) shows that the Bethe state $\Psi(v)$ is an eigenstate of the transfer matrix $t(u)$ under the condition $F_\alpha \equiv 0$ ($\alpha = 1, \ldots, M$), i.e.

$$\prod_{\beta\neq\alpha}^{M} \frac{\operatorname{sh}(v_\alpha - v_\beta - \eta)\operatorname{sh}(v_\alpha + v_\beta)}{\operatorname{sh}(v_\alpha - v_\beta + \eta)\operatorname{sh}(v_\alpha + v_\beta + 2\eta)} = \frac{\operatorname{sh}(v_\alpha - \xi_- + \eta)\operatorname{sh}(v_\alpha - \xi_+ + \eta)}{\operatorname{sh}(v_\alpha + \xi_-)\operatorname{sh}(v_\alpha + \xi_+)}\left[\prod_{k}^{N} \frac{\operatorname{sh}(v_\alpha - z_k)\operatorname{sh}(v_\alpha + z_k)}{\operatorname{sh}(v_\alpha - z_k + \eta)\operatorname{sh}(v_\alpha + z_k + \eta)}\right]. \tag{2.35}$$

This equation with $z_k \equiv 0$ corresponds to the Bethe ansatz equation (BAE) for the spin-1/2 XXZ spin chain with boundary derived in [1, 5].

3. The Gaudin magnet

We show that the Gaudin magnet with boundary can be derived from identity (2.31). The Gaudin magnet introduced in [10] was given by taking the quasi-classical limit $\eta \to 0$ of the transfer matrix $\operatorname{Tr} T(u)$ for the inhomogeneous spin chain [13]. This fact indicates that the Hamiltonian is written in terms of the solution of the classical YBE. In our open-boundary Gaudin magnet we require the constraint for the parameters $\xi_\pm$:

$$\xi_+ = -\xi_- \equiv \xi. \tag{3.1}$$

Due to the quasi-classical condition (2.5), we have the power series expansion around the point $\eta = 0$ for each term in (2.31):

$$t(u = z_j) = \operatorname{sh}(z_j + \xi)\operatorname{sh}(z_j - \xi)(1 + \eta H_j + \mathrm{O}(\eta^2)) \tag{3.2}$$

$$\Lambda(u = z_j) = \operatorname{sh}(z_j + \xi)\operatorname{sh}(z_j - \xi)(1 + \eta E_j + \mathrm{O}(\eta^2)) \tag{3.3}$$

$$F_\alpha = \eta^2 \frac{-\operatorname{sh}(2z_j)\operatorname{sh}(v_\alpha + \xi)\operatorname{sh}(v_\alpha - \xi)}{\operatorname{sh}(z_j - v_\alpha)\operatorname{sh}(z_j + v_\alpha)} f_\alpha + \mathrm{O}(\eta^3) \tag{3.4}$$

where the 'Hamiltonian' H_j, 'energy' E_j, and 'unwanted factor' f_α are calculated as

$$\begin{aligned} H_j = {} & \frac{\operatorname{sh}(2z_j) - \operatorname{sh}(2\xi)\sigma_j^z}{2\operatorname{sh}(z_j + \xi)\operatorname{sh}(z_j - \xi)} + \sum_{k}^{N} \frac{1}{\operatorname{sh}(z_j + z_k)} \\ & \times \left(\frac{\operatorname{sh}(\xi + z_j)}{\operatorname{sh}(\xi - z_j)}\sigma_j^-\sigma_k^+ + \frac{\operatorname{sh}(\xi - z_j)}{\operatorname{sh}(\xi + z_j)}\sigma_j^+\sigma_k^- + \operatorname{ch}(z_j + z_k)\frac{\sigma_j^z\sigma_k^z - 1}{2}\right) \\ & + \sum_{k\neq j}^{N} \frac{1}{\operatorname{sh}(z_j - z_k)}\left(\sigma_j^+\sigma_k^- + \sigma_j^-\sigma_k^+ + \operatorname{ch}(z_j - z_k)\frac{\sigma_j^z\sigma_k^z - 1}{2}\right) \end{aligned} \tag{3.5}$$

$$E_j = \frac{\operatorname{ch}(2z_j)}{\operatorname{sh}(2z_j)} - \sum_{\alpha}^{M}\left(\frac{\operatorname{ch}(z_j - v_\alpha)}{\operatorname{sh}(z_j - v_\alpha)} + \frac{\operatorname{ch}(z_j + v_\alpha)}{\operatorname{sh}(z_j + v_\alpha)}\right) \tag{3.6}$$

$$\begin{aligned} f_\alpha = {} & \frac{\operatorname{ch}(v_\alpha + \xi)}{\operatorname{sh}(v_\alpha + \xi)} + \frac{\operatorname{ch}(v_\alpha - \xi)}{\operatorname{sh}(v_\alpha - \xi)} + 2\sum_{\beta\neq\alpha}^{M}\left(\frac{\operatorname{ch}(v_\alpha - v_\beta)}{\operatorname{sh}(v_\alpha - v_\beta)} + \frac{\operatorname{ch}(v_\alpha + v_\beta)}{\operatorname{sh}(v_\alpha + v_\beta)}\right) \\ & - \sum_{k}^{N}\left(\frac{\operatorname{ch}(v_\alpha - z_k)}{\operatorname{sh}(v_\alpha - z_k)} + \frac{\operatorname{ch}(v_\alpha + z_k)}{\operatorname{sh}(v_\alpha + z_k)}\right). \end{aligned} \tag{3.7}$$

We remark that due to constraint (3.1) the first term in $t(u = z_j)$ becomes a c-number. This proves the integrability of the Hamiltonian H_j,

$$[H_j, H_k] = 0 \qquad \text{for } j = 1, \ldots, N \tag{3.8}$$

which is obtained from the commutativity of the transfer matrix (2.20). The Bethe states, Ψ and Ψ_α, can also be expanded as

$$\Psi(v) = \eta^M \phi + O(\eta^{M+1}) \tag{3.9}$$

$$\Psi_\alpha(z_j) = \eta^{M-1}(-\mathrm{sh}(z_j+\xi))\sigma_j^- \phi_\alpha + O(\eta^M) \tag{3.10}$$

with

$$\phi = \prod_\alpha^M \left(\sum_k^N \left(-\frac{\mathrm{sh}(v_\alpha+\xi)}{\mathrm{sh}(v_\alpha - z_k)} + \frac{\mathrm{sh}(v_\alpha-\xi)}{\mathrm{sh}(v_\alpha+z_k)} \right) \sigma_k^- \right) |\Omega\rangle \tag{3.11}$$

$$\phi_\alpha = \prod_{\beta\neq\alpha}^M \left(\sum_k^N \left(-\frac{\mathrm{sh}(v_\beta+\xi)}{\mathrm{sh}(v_\beta - z_k)} + \frac{\mathrm{sh}(v_\beta-\xi)}{\mathrm{sh}(v_\beta+z_k)} \right) \sigma_k^- \right) |\Omega\rangle. \tag{3.12}$$

When we combine the terms proportional to η^{M+1} in (2.31), we obtain the so-called 'off-shell Bethe ansatz' equation

$$H_j\phi = E_j\phi + \sum_\alpha^M \frac{\mathrm{sh}(2z_j)\,\mathrm{sh}(v_\alpha-\xi)\,\mathrm{sh}(v_\alpha+\xi)}{\mathrm{sh}(z_j-v_\alpha)\,\mathrm{sh}(z_j+v_\alpha)\,\mathrm{sh}(z_j-\xi)} f_\alpha \sigma_j^- \phi_\alpha. \tag{3.13}$$

This equation suggests that the Bethe state ϕ (3.11) is an eigenstate of the Gaudin's Hamiltonian H_j, iff a set of rapidities $\{v_\alpha\}$ is set to satisfy $f_\alpha \equiv 0$ ($\alpha = 1, \ldots, M$), i.e.

$$\sum_k^N \left(\frac{\mathrm{ch}(v_\alpha - z_k)}{\mathrm{sh}(v_\alpha - z_k)} + \frac{\mathrm{ch}(v_\alpha + z_k)}{\mathrm{sh}(v_\alpha + z_k)} \right) = \frac{\mathrm{ch}(v_\alpha+\xi)}{\mathrm{sh}(v_\alpha+\xi)} + \frac{\mathrm{ch}(v_\alpha-\xi)}{\mathrm{sh}(v_\alpha-\xi)} + 2\sum_{\beta\neq\alpha}^M \left(\frac{\mathrm{ch}(v_\alpha - v_\beta)}{\mathrm{sh}(v_\alpha - v_\beta)} + \frac{\mathrm{ch}(v_\alpha + v_\beta)}{\mathrm{sh}(v_\alpha + v_\beta)} \right). \tag{3.14}$$

We have derived the eigenstate and the energy of the XXZ-type Gaudin magnet with boundary. We note that in the rational limit the Hamiltonian H_j (3.5) reduces to the form

$$H_j = \sum_{k\neq j}^N \left(\frac{P_{jk}-1}{z_j-z_k} + \frac{\xi - z_j\sigma_j^z}{\xi - z_j}\frac{P_{jk}-1}{z_j+z_k}\frac{\xi+z_j\sigma_j^z}{\xi+z_j} \right) + \frac{z_j - \xi\sigma_j^z}{z_j^2-\xi^2} \tag{3.15}$$

where P_{jk} is a permutation operator in spin space:

$$P_{jk} = \sigma_j^+ \otimes \sigma_k^- + \sigma_j^- \otimes \sigma_k^+ + \frac{\sigma_j^z \otimes \sigma_k^z + 1}{2}. \tag{3.16}$$

4. The Knizhnik–Zamolodchikov equation

We consider the KZ-type differential equation

$$\nabla_j \psi = 0 \qquad \text{for } j = 1, 2, \ldots, N. \tag{4.1}$$

where the differential operator ∇_j is defined by use of Gaudin's Hamiltonian H_j (3.5) (and its rational limit (3.15)):

$$\nabla_j = \kappa \frac{\partial}{\partial z_j} - H_j. \tag{4.2}$$

We remark that κ is an arbitrary parameter. The integrable condition for a set of the KZ-type differential operators ∇_j,

$$[\nabla_j, \nabla_k] = 0 \qquad \text{for } j, k = 1, \ldots, N \tag{4.3}$$

is satisfied for the case

$$\frac{\partial H_j}{\partial z_k} = \frac{\partial H_k}{\partial z_j}.$$

Then the parameter ξ is solved as

$$\xi \equiv 0. \tag{4.4}$$

In this case the differential operator ∇_j coincides with the B-type KZ differential operator considered in [33]. Gaudin's Hamiltonians H_j include an arbitrary parameter ξ, which disappears in the family of the mutually commuting B-type KZ differential operators ∇_j. We rewrite the explicit forms of the operator H_j and the Bethe state ϕ in the limit $\xi \to 0$ as

$$\begin{aligned} H_j = \frac{\operatorname{ch} z_j}{\operatorname{sh} z_j} &+ \sum_{k\neq j}^{N} \frac{1}{\operatorname{sh}(z_j - z_k)} \left(\sigma_j^+ \sigma_k^- + \sigma_j^- \sigma_k^+ + \operatorname{ch}(z_j - z_k) \frac{\sigma_j^z \sigma_k^z - 1}{2} \right) \\ &+ \sum_{k}^{N} \frac{1}{\operatorname{sh}(z_j + z_k)} \sigma_j^z \left(\sigma_j^- \sigma_k^+ + \sigma_j^+ \sigma_k^- + \operatorname{ch}(z_j + z_k) \frac{\sigma_j^z \sigma_k^z - 1}{2} \right) \sigma_j^z \end{aligned} \tag{4.5}$$

$$\phi(z, v) = \prod_{\alpha}^{M} \left\{ \sum_{k}^{N} \left(- \frac{\operatorname{sh} v_\alpha}{\operatorname{sh}(v_\alpha - z_k)} + \frac{\operatorname{sh} v_\alpha}{\operatorname{sh}(v_\alpha + z_k)} \right) \sigma_k^- \right\} |\Omega\rangle. \tag{4.6}$$

Following the idea of [19,21], we define the hypergeometric function $\chi(z, v)$ by a set of differential equations

$$\kappa \frac{\partial \chi}{\partial z_j} = E_j \chi \qquad \text{for } j = 1, \ldots, N \tag{4.7a}$$

$$\kappa \frac{\partial \chi}{\partial v_\alpha} = f_\alpha \chi \qquad \text{for } \alpha = 1, \ldots, M. \tag{4.7b}$$

The integrability of these differential equations follows from the conditions

$$\frac{\partial E_j}{\partial z_k} = \frac{\partial E_k}{\partial z_j} \qquad \frac{\partial E_j}{\partial v_\alpha} = \frac{\partial f_\alpha}{\partial z_j} \qquad \frac{\partial f_\alpha}{\partial v_\beta} = \frac{\partial f_\beta}{\partial v_\alpha}.$$

In fact, it is straightforward to solve the differential equations (4.7*a*); its solution $\chi(z, v)$ is a hypergeometric function

$$\begin{aligned} \chi(z, v) = \prod_{j}^{N} (\operatorname{sh}(2z_j))^{1/2\kappa} \prod_{j}^{N} \prod_{\alpha}^{M} (\operatorname{sh}(z_j - v_\alpha) \operatorname{sh}(z_j + v_\alpha))^{-1/\kappa} \prod_{\alpha}^{M} (\operatorname{sh} v_\alpha)^{2/\kappa} \\ \times \prod_{\alpha<\beta}^{M} (\operatorname{sh}(v_\alpha - v_\beta) \operatorname{sh}(v_\alpha + v_\beta))^{2/\kappa}. \end{aligned} \tag{4.8}$$

One can introduce the wavefunction $\psi(z)$ in the integrated form, which has a hypergeometric kernel, as

$$\psi(z) = \oint_C \prod_{\alpha}^{M} \frac{\mathrm{d}v_\alpha}{\operatorname{sh} v_\alpha} \chi(z, v) \phi(z, v). \tag{4.9}$$

The integration path C is taken over a closed contour in the Riemann surface such that the integrand resumes its initial value after v_α has described it. The integral function $\psi(z)$ is in fact a solution of the B-type KZ equation (4.1):

$$\nabla_j \psi(z) = 0.$$

To prove (4.1) we use the fact that the Bethe state $\phi(z, v)$ (4.6) satisfies

$$\frac{\partial\phi}{\partial z_j} = \sum_{\alpha}^{M} \operatorname{sh}(2v_\alpha) \operatorname{ch} z_j \frac{\operatorname{sh}(z_j - v_\alpha)\operatorname{sh}(z_j + v_\alpha) - 2\operatorname{sh}^2 z_j}{\operatorname{sh}^2(z_j - v_\alpha)\operatorname{sh}^2(z_j + v_\alpha)} \sigma_j^- \phi_\alpha$$

where ϕ_α is defined in (3.12) with the condition $\xi \equiv 0$. One sees that the function ϕ_α does not depend on v_α. Then equality (4.1) can be proved simply as

$$\begin{aligned}
\kappa\frac{\partial}{\partial z_j}\psi(z) &= \oint_C \prod_{\alpha}^{M} \frac{dv_\alpha}{\operatorname{sh} v_\alpha}\left(\kappa\frac{\partial\chi}{\partial z_j}\phi + \kappa\chi\frac{\partial\phi}{\partial z_j}\right) \\
&= \oint_C \prod_{\alpha}^{M} \frac{dv_\alpha}{\operatorname{sh} v_\alpha}\left(E_j\chi\phi + \kappa\chi\frac{\partial\phi}{\partial z_j}\right) \\
&= \oint_C \prod_{\alpha}^{M} \frac{dv_\alpha}{\operatorname{sh} v_\alpha}\left\{H_j\chi\phi - \kappa\sum_{\alpha}^{M}\frac{2\operatorname{ch} z_j \operatorname{sh}^2 v_\alpha}{\operatorname{sh}(z_j - v_\alpha)\operatorname{sh}(z_j + v_\alpha)}\frac{\partial\chi}{\partial v_\alpha}\sigma_j^-\phi_\alpha\right. \\
&\quad \left. - \kappa\sum_{\alpha}^{M}\operatorname{sh} v_\alpha\left(\frac{\operatorname{ch}(v_\alpha - z_j)}{\operatorname{sh}^2(v_\alpha - z_j)} + \frac{\operatorname{ch}(v_\alpha + z_j)}{\operatorname{sh}^2(v_\alpha + z_j)}\right)\chi\sigma_j^-\phi_\alpha\right\} \\
&= H_j\psi - \kappa\sum_{\alpha}^{M}\oint_C\left[\prod_{\beta\neq\alpha}^{M}\frac{dv_\beta}{\operatorname{sh} v_\beta}\right]dv_\alpha\frac{\partial}{\partial v_\alpha}\left(\chi\frac{2\operatorname{ch} z_j \operatorname{sh} v_\alpha}{\operatorname{sh}(z_j - v_\alpha)\operatorname{sh}(z_j + v_\alpha)}\right)\sigma_j^-\phi_\alpha \\
&= H_j\psi.
\end{aligned}$$

5. Discussion

We have constructed the integral representation for the solution of the B-type KZ equation. We summarize our result for the rational case. In this case the integrable condition for the KZ-type differential operators, $[\nabla_j, \nabla_k] = 0$, is satisfied for two cases: $\xi = 0$ and $\xi = \infty$. The first case gives the B-type KZ equation

$$\kappa\frac{\partial}{\partial z_j}\psi^{\mathrm{B}}(z) = \left\{\sum_{k\neq j}^{N}\left(\frac{P_{jk} - 1}{z_j - z_k} + \frac{\overline{P}_{jk} - 1}{z_j + z_k}\right) + \frac{1}{z_j}\right\}\psi^{\mathrm{B}}(z) \tag{5.1}$$

where $\overline{P}_{jk} \equiv \sigma_j^z\sigma_k^z P_{jk}$. The integral solution can be explicitly written as

$$\begin{aligned}
\psi^{\mathrm{B}}(z) = \oint_C dv \prod_{j}^{N} z_j^{1/2\kappa} \prod_{j}^{N}\prod_{\alpha}^{M}(z_j^2 - v_\alpha^2)^{-1/\kappa}\prod_{\alpha}^{M} v_\alpha^{2/\kappa}\prod_{\alpha<\beta}^{M}(v_\alpha^2 - v_\beta^2)^{2/\kappa} \\
\times\prod_{\alpha}^{M}\left(\sum_{k}^{N}\frac{z_k}{z_k^2 - v_\alpha^2}\sigma_k^-\right)|\Omega\rangle.
\end{aligned} \tag{5.2}$$

On the other hand, one can see that the second case, $\xi = \infty$, corresponds to the A-type KZ equation

$$\kappa\frac{\partial}{\partial z_j}\psi^{\mathrm{A}}(z) = \sum_{k\neq j}^{N}\frac{P_{jk} - 1}{z_j - z_k}\psi^{\mathrm{A}}(z) \tag{5.3}$$

and that the integral representation for the solution is given by

$$\psi^{\mathrm{A}}(z) = \oint_C dv \prod_{\alpha}^{M}\prod_{j}^{N}(v_\alpha - z_j)^{-1/\kappa}\prod_{\alpha<\beta}^{M}(v_\alpha - v_\beta)^{2/\kappa}\prod_{\alpha}^{M}\left(\sum_{k}^{N}\frac{1}{z_k - v_\alpha}\sigma_k^-\right)|\Omega\rangle. \tag{5.4}$$

From this fact, one can conclude that the rational 'off-shell Bethe ansatz equation' (3.13) for the boundary Gaudin magnet intertwines the A- and B-type KZ equations.

We only give the integral representation for the solution of the spin-$1/2$ B-type KZ equation. The generalization to the su(n) B-type KZ equation should be done from the view point of the Gaudin magnet with boundary [34].

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